Quasi-isotropic solution of the Einstein equations near a cosmological singularity for a two-fluid cosmological model

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Abstract. The quasi-isotropic inhomogeneous solution of the Einstein equations near a cosmological singularity in the form of a series expansion in the synchronous system of reference, first found by Lifshitz and Khalatnikov in 1960, is generalized to the case of a two-fluid cosmological model. This solution describes non-decreasing modes of adiabatic and isocurvature scalar perturbations and gravitational waves in the regime when deviations of a space-time metric from the homogeneous isotropic Friedmann-Robertson-Walker (FRW) background are large while locally measurable quantities like Riemann tensor components are still close to their FRW values. The general structure of the perturbation series is presented and the first coefficients of the series expansion for the metric tensor and the fluid energy densities and velocities are calculated explicitly.

1. Introduction

In the recent paper [1], we presented the generalization of the quasi-isotropic solution of the Einstein equations near a cosmological singularity found by Lifshitz and Khalatnikov [2] for the Universe filled by radiation with the equation of state $p = \frac{\varepsilon}{3}$ to the case of an arbitrary one-fluid cosmological model. Here, this solution is further generalized to the case of the Universe filled by two ideal barotropic fluids.

As is well known, modern cosmology deals with many very different types of matter. In comparison with the old standard model of the hot Universe (the Big Bang), the situation has been dramatically changed, first, with the development of inflationary cosmological models which contain an inflaton effective scalar field or/and other exotic types of matter as an important ingredient [3], and second, with the understanding that the main part of the non-relativistic matter in the present Universe is non-baryonic – cold dark matter (CDM). Furthermore, the appearance of brane and M-theory cosmological models [4] and the discovery of the cosmic acceleration [5] (see also [6] for a review) suggests that matter playing an essential role at different stages of cosmological evolution is multi-component generically, and these components may obey very different equations of state. Moreover, the very notion of the equation of state appears to be not fundamental; it has only a limited range of validity as compared to a more fundamental field-theoretical description. From this general point of view, the generalization of the quasi-isotropic solution to the case of two ideal barotropic fluids with constant but different p/ε ratios seems to be a natural and important next logical step. The most straightforward candidates for these fluids are non-relativistic matter (CDM) with $p = 0$ and radiation. One more popular candidate is ultra-stiff matter $(p = \varepsilon)$ which underlying physical model is a minimally coupled to gravity scalar field in the regime when its potential energy may be neglected as compared to its kinetic energy [7] (in particular, it may be the inflaton field itself after the end of inflation).

To explain the physical sense of the quasi-isotropic solution, let us remind that it represents the most generic spatially inhomogeneous generalization of the FRW space-time in which the space-time is locally FRW-like near the cosmological singularity $t = 0$ (in particular, its Weyl tensor is much less than its Riemann tensor). On the other

hand, generically it is very inhomogeneous globally and may have a very complicated spatial topology. As was shown in [8, 1] (see also [9]), such a solution contains 3 arbitrary functions of space coordinates. From the FRW point of view, these 3 degrees of freedom represent the growing (non-decreasing in terms of metric perturbations) mode of adiabatic perturbations and the non-decreasing mode of gravitational waves (with two polarizations) in the case when deviations of a space-time metric from the FRW one are not small. So, the quasi-isotropic solution is not a generic solution of the Einstein equations with a barotropic fluid. Therefore, one should not expect this solution to arise in the course of generic gravitational collapse (in particular, inside a black hole event horizon). The generic solution near a space-like curvature singularity (for $p < \varepsilon$) has a completely different structure consisting of the infinite sequence of anisotropic vacuum Kasner-like eras with space-dependent Kasner exponents [10].

For this reason, the quasi-isotropic solution had not attracted much interest for about twenty years. Its new life began after the development of successful inflationary models (i.e., with "graceful exit" from inflation) and the theory of generation of perturbations during inflation, because it had immediately become clear that generically (without fine tuning of initial conditions) scalar metric perturbations after the end of inflation remained small in a finite region of space which was much less than the whole causally connected space volume produced by inflation. It appears that the quasi-isotropic solution can be used for a global description of a part of space-time after inflation which belongs to "one post-inflationary universe". The latter is defined as a connected part of space-time where the hyper-surface $t = t_f(\mathbf{r})$ describing the moment when inflation ends is space-like and, therefore, can be made the surface of constant (zero) synchronous time by a coordinate transformation. This directly follows from the derivation of perturbations generated during inflation given in [11] (see Eq. (17) of that paper) which is valid in case of large perturbations, too. Thus, when used in this context, the quasi-isotropic solution represents an *intermediate* asymptotic regime during expansion of the Universe after inflation. The synchronous time t appearing in it is the proper time since the *end* of inflation, and the region of validity of the solution is from $t = 0$ up to a moment in future when spatial gradients become important. For sufficiently large scales, the latter moment may be rather late, even of the order or larger than the present age of the Universe. Note also the analogue of the quasi-isotropic solution *before* the end of inflation is given by the generic quasi-de Sitter solution found in [12]. Both solutions can be smoothly matched across the hypersurface of the end of inflation.

Generalization of the quasi-isotropic solution to the case of multiple fluids provides a possibility to investigate the growing isocurvature mode of scalar perturbations and its effect on the growing mode of adiabatic perturbations in the regime when a space-time metric *may not* be globally represented as the FRW metric with small perturbations. Isocurvature perturbation of the energy density of a subdominant fluid need not be small, too. Note that the splitting of scalar perturbations into the adiabatic and the isocurvature mode is, to some extent, conventional (especially, in models with decay

of one forms of matter into others, e.g., in the so called "curvaton" model [13]). Of course, this does not affect theoretical predictions for any observable quantity since contributions from all modes should be taken into account. We use the most natural splitting which is essentially the same as in the standard cosmological (Λ)CDM + radiation model, namely, that metric perturbations for the growing isocurvature mode are zero at $t = 0$. However, they are not zero for $t > 0$ and grow with time.

It is well known that the growing adiabatic mode remains constant in the synchronous reference system in the leading long-wave approximation for any one-fluid (and even any one-component matter) model both in the linear regime and with all back reaction effects taken into account. This immediately follows from the very form of the quasi-isotropic solution for one fluid – it corresponds to the *factorization* of spatial and temporal variables in its leading term. In a multi-component case, this property is known to be valid in the linear approximation, namely, that there always exists one solution of equations for perturbations (which we just call the growing adiabatic mode) which remains constant in the long-wave regime (see, e.g., [14] and for a more recent discussion [15]). However, some concern was expressed (e.g., in [16]) if this remains valid in the long-wave non-linear multi-component regime. We will show that the factorization of variables in the first term occurs in the generalized quasi-isotropic solution, too, and this term remains the leading one either until spatial gradients become important, or until the energy density of a firstly subdominant fluid becomes comparable to that of the dominant one (that results in the change of the expansion law).

2. Series expansion of the quasi-isotropic solution for a two-fluid case

Let us consider a FRW model with two barotropic fluids satisfying the equations of state:

$$p_l = k_l \varepsilon_l, \quad l = 1, 2 \tag{1}$$

where p_l and ε_l are the pressure and the energy density of the corresponding fluid. For usual fluids, $0 \le k_l \le 1$. However, similar to the case of one fluid, our solution remains valid for a larger range $k_l > -\frac{1}{3}$ for both fluids. Let us take

$$k_2 > k_1. \tag{2}$$

As usually [8, 17], we will work in the synchronous system of reference

$$ds^2 = dt^2 - \gamma_{\alpha\beta}dx^\alpha dx^\beta. \tag{3}$$

Though this system of reference is not fixed uniquely (in other words, some of arbitrary functions entering into a solution written in this system are fictitious, or gauge artifacts), the choice of the initial cosmological singularity at the hypersurface $t = 0$ in the quasi-isotropic solution (see Eq. (20) below) removes half of this ambiguity leaving only 3D spatial rotations in the leading term (with corresponding terms in higher orders) as the remaining gauge freedom.

To find the time dependence of the leading term of the quasi-isotropic solution, we write down the FRW energy equation in the vicinity of the cosmological singularity which reads

$$\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \frac{A_1}{a^{\alpha_1}} + \frac{A_2}{a^{\alpha_2}} \tag{4}$$

where A_1 and A_2 are some constants, while the exponents α_1 and α_2 are defined as

$$\alpha_1 = 3(1 + k_1), \tag{5}$$

$$\alpha_2 = 3(1 + k_2) > \alpha_1. \tag{6}$$

We will look for a time dependence of the cosmological scale factor $a(t)$ (which determines the determinant of the spatial metric $\gamma_{\alpha\beta}$) near the singularity in the following form:

$$a(t) = \sum_{n=0} a_n t^{\gamma_n}. \tag{7}$$

Substituting this expansion into the Friedmann equation (4) and comparing the smallest (i.e., most singular) powers in t, one can easily find that

$$\gamma_0 = \frac{2}{\alpha_2}. \tag{8}$$

Thus, the lowest exponent in the quasi-isotropic solution is defined by the stiffer fluid. The next term in the expansion (7) arises due to the presence of the second fluid in the Universe. Its time dependence is characterized by the exponent

$$\gamma_1 = 2 + \gamma_0(1 - \alpha_1) = 2 + \frac{2}{\alpha_2} - \frac{2\alpha_1}{\alpha_2}. \tag{9}$$

The third term in the expansion (7) arises due to the presence of the curvature term. It has the same exponent as in the case of the one - fluid cosmological model [1]:

$$\gamma_2 = 2 - \gamma_0 = 2 - \frac{2}{\alpha_2}. \tag{10}$$

The next term of the expansion is again connected with the presence of the second fluid. The correspondent exponent is equal to

$$\gamma_3 = 4 + \gamma_0 - 2\gamma_0\alpha_1 = 4 + \frac{2}{\alpha_2} - \frac{4\alpha_1}{\alpha_2}. \tag{11}$$

Notice that three cases are possible depending on relation between α_1 and α_2:

$$1) \quad \gamma_3 > \gamma_2, \quad \alpha_2 > 2\alpha_1 - 2 , \tag{12}$$

e.g., if $k_2 = 1$, $\alpha_2 = 6$ and $k_1 = 0$, $\alpha_1 = 3$, i.e., the dominant fluid is ultra-stiff matter and the subdominant fluid is dust;

$$2) \quad \gamma_3 = \gamma_2, \quad \alpha_2 = 2\alpha_1 - 2 , \tag{13}$$

e.g., in the case $k_2 = 1$, $\alpha_2 = 6$ and $k_1 = 1/3$, $\alpha_1 = 4$, so the first fluid is ultra-stiff matter and the second fluid is radiation, or in the dust–radiation case $k_2 = 1/3$, $\alpha_2 = 4$ and $k_1 = 0$, $\alpha_1 = 3$;

$$3) \quad \gamma_3 < \gamma_2, \quad \alpha_2 < 2\alpha_1 - 2 , \tag{14}$$

(e.g., in the case $k_2 = 1$, $\alpha_2 = 6$ and $k_1 = 2/3$, $\alpha_1 = 5$).

Now it is easy to understand that all terms which are present in the series expansion for the one-fluid quasi-isotropic solution remain in the solution for a two-fluid case because the mechanism of their appearance remains the same: interference between the dominant fluid and the curvature term. It is convenient to write a general formula for this set of terms in the following form:

$$\gamma_{n,0} = \gamma_0 + n(2 - 2\gamma_0), \ n = 0, 1, \dots. \tag{15}$$

Another sequence of terms in the quasi-isotropic solution arises due to interference between the two fluids. It has the following set of exponents:

$$\gamma_{0,m} = \gamma_0 + m(2 - \gamma_0\alpha_1), \ n = 1, \dots. \tag{16}$$

In addition, the full set of terms occurring in the quasi-isotropic solution includes terms arising from mixing between the two sequences (15) and (16). Thus, the expansion (7) can be represented as

$$a(t) = \sum_{n,m=0} a_{n,m} t^{\gamma_{n,m}} \tag{17}$$

where

$$\gamma_{n,m} = \gamma_0 + n(2 - 2\gamma_0) + m(2 - \gamma_0\alpha_1). \tag{18}$$

It is easy to see that if the number

$$\frac{2 - 2\gamma_0}{2 - \gamma_0\alpha_1} = \frac{\alpha_2 - 2}{\alpha_2 - \alpha_1} = \frac{3k_2 + 1}{3(k_2 - k_1)} \tag{19}$$

is rational, a degeneracy between exponents $\gamma_{n,m}$ in Eq. (18) is possible (an example of such degeneracy is given in Eq. (13)). However, it does not create any peculiarities in the structure of the quasi-isotropic solution. It is important that the expansion (17) with the exponents given in Eq. (18) contains all possible cross-terms.

Correspondingly, the inhomogeneous quasi-isotropic solution for the spatial metric $\gamma_{\alpha\beta}$ can be represented as

$$\gamma_{\alpha\beta} = \sum_{n,m=0} \gamma_{\alpha\beta}^{(n,m)} t^{2\gamma_0 + n(2-2\gamma_0) + m(2-\gamma_0\alpha_1)}. \tag{20}$$

For illustrative purposes, let us display the structure of this solution for some simple two-fluid models. For the model with radiation and dust, the series (20) can be rewritten as

$$\gamma_{\alpha\beta} = \sum_{n=0} \gamma_{\alpha\beta}^{(n)} t^{1+n/2} . \tag{21}$$

For ultra-stiff matter and radiation, one has

$$\gamma_{\alpha\beta} = \sum_{n=0} \gamma_{\alpha\beta}^{(n)} t^{2/3+2n/3} . \tag{22}$$

In the case of ultra-stiff matter and dust, we get

$$\gamma_{\alpha\beta} = \gamma_{\alpha\beta}^{(0)} t^{2/3} + \gamma_{\alpha\beta}^{(1)} t^{5/3} + \gamma_{\alpha\beta}^{(2)} t^2 + \sum_{n=3} \gamma_{\alpha\beta}^{(n)} t^{8/3+(n-3)/3} , \tag{23}$$

and for stiff matter and the fluid with $k_1 = 2/3$, the solution reads

$$\gamma_{\alpha\beta} = \sum_{n=0} \gamma_{\alpha\beta}^{(n)} t^{2/3+n/3} \ . \tag{24}$$

Now, we will deduce explicit formulae for the first terms of the quasi-isotropic solution. The following notations will be used:

$$\gamma_{\alpha\beta} = a_{\alpha\beta} t^{2\gamma_0} + b_{\alpha\beta} t^{2\gamma_0 + 2 - \gamma_0 \alpha_1} + c_{\alpha\beta} t^2 + d_{\alpha\beta} t^{2\gamma_0 + 4 - 2\gamma_0 \alpha_1} + \cdots . \tag{25}$$

As usually, the spatial metric $a_{\alpha\beta}$ is arbitrary. For the time being we treat the metric $b_{\alpha\beta}$ in the second term of the expansion (25) arising due to the presence of the second fluid as an arbitrary positively defined symmetric tensor.

The inverse spatial metric reads

$$\gamma^{\alpha\beta} = a^{\alpha\beta} t^{-2\gamma_0} - b^{\alpha\beta} t^{2 - \gamma_0 \alpha_1 - 2\gamma_0}$$
$$- c^{\alpha\beta} t^{2 - 4\gamma_0} - d^{\alpha\beta} t^{4 - 2\gamma_0 \alpha_1 - 2\gamma_0} + b_\delta^\alpha b^{\delta\beta} t^{4 - 2\gamma_0 \alpha_1 - 2\gamma_0} \tag{26}$$

where $a^{\alpha\beta}$ is defined by the relation

$$a^{\alpha\beta} a_{\beta\gamma} = \delta_\gamma^\alpha, \tag{27}$$

while the indices of all other matrices are lowered and raised by $a_{\alpha\beta}$ and $a^{\alpha\beta}$, for example,

$$b_\beta^\alpha = a^{\alpha\gamma} b_{\gamma\beta}. \tag{28}$$

Let us also write down the expressions for the extrinsic curvature, its contractions and its derivatives:

$$\kappa_{\alpha\beta} \equiv \frac{\partial \gamma_{\alpha\beta}}{\partial t} = 2\gamma_0 a_{\alpha\beta} t^{2\gamma_0 - 1} + (2\gamma_0 + 2 - \gamma_0 \alpha_1) b_{\alpha\beta} t^{2\gamma_0 + 1 - \gamma_0 \alpha_1}$$
$$+ 2c_{\alpha\beta} t + (2\gamma_0 + 4 - 2\gamma_0 \alpha_1) d_{\alpha\beta} t^{2\gamma_0 + 3 - 2\gamma_0 \alpha_1}, \tag{29}$$

$$\kappa_\alpha^\beta = 2\gamma_0 \delta_\alpha^\beta t^{-1} + (2 - \gamma_0 \alpha_1) b_\alpha^\beta t^{1 - \gamma_0 \alpha_1} + 2(1 - \gamma_0) c_\alpha^\beta t^{1 - 2\gamma_0}$$
$$+ 2(2 - \gamma_0 \alpha_1) d_\alpha^\beta t^{3 - 2\gamma_0 \alpha_1} - (2 - \gamma_0 \alpha_1) b_{\alpha\delta} b^{\delta\beta} t^{3 - 2\gamma_0 \alpha_1}, \tag{30}$$

$$\kappa_\alpha^\alpha = 6\gamma_0 t^{-1} + (2 - \gamma_0 \alpha_1) b t^{1 - \gamma_0 \alpha_1} + 2(1 - \gamma_0) c_\alpha^\beta t^{1 - 2\gamma_0}$$
$$+ 2(2 - \gamma_0 \alpha_1) d t^{3 - 2\gamma_0 \alpha_1} - (2 - \gamma_0 \alpha_1) b_{\alpha\beta} b^{\alpha\beta} t^{3 - 2\gamma_0 \alpha_1}, \tag{31}$$

$$\frac{\partial \kappa_\alpha^\beta}{\partial t} = -2\gamma_0 \delta_\alpha^\beta t^{-2} + (2 - \gamma_0 \alpha_1)(1 - \gamma_0 \alpha_1) b_\alpha^\beta t^{-\gamma_0 \alpha_1}$$
$$+ 2(1 - \gamma_0)(1 - 2\gamma_0) c_\alpha^\beta t^{-2\gamma_0} + 2(2 - \gamma_0 \alpha_1)(3 - 2\gamma_0 \alpha_1) d_\alpha^\beta t^{2 - 2\gamma_0 \alpha_1}$$
$$- (2 - \gamma_0 \alpha_1)(3 - 2\gamma_0 \alpha_1) b_{\alpha\delta} b^{\delta\beta} t^{2 - 2\gamma_0 \alpha_1}, \tag{32}$$

$$\frac{\partial \kappa_\alpha^\alpha}{\partial t} = -6\gamma_0 t^{-2} + (2 - \gamma_0 \alpha_1)(1 - \gamma_0 \alpha_1) b t^{-\gamma_0 \alpha_1}$$
$$+ 2(1 - \gamma_0)(1 - 2\gamma_0) c t^{-2\gamma_0} + 2(2 - \gamma_0 \alpha_1)(3 - 2\gamma_0 \alpha_1) d t^{2 - 2\gamma_0 \alpha_1}$$
$$- (2 - \gamma_0 \alpha_1)(3 - 2\gamma_0 \alpha_1) b_{\alpha\delta} b^{\delta\alpha} t^{2 - 2\gamma_0 \alpha_1}, \tag{33}$$

$$\kappa_\alpha^\beta \kappa_\beta^\alpha = 12\gamma_0^2 t^{-2} + 4\gamma_0 (2 - \gamma_0 \alpha_1) b t^{-\gamma_0 \alpha_1}$$
$$+ 8\gamma_0 (1 - \gamma_0) c t^{-2\gamma_0} + 8\gamma_0 (2 - \gamma_0 \alpha_1) d t^{2 - 2\gamma_0 \alpha_1}$$
$$+ (2 - \gamma_0 \alpha_1)(2 - \gamma_0 \alpha_1 - 4\gamma_0) b_\alpha^\beta b_\beta^\alpha t^{2 - 2\gamma_0 \alpha_1}. \tag{34}$$

Also, we need an explicit expression for the determinant of the spatial metric and its time derivative:

$$\gamma \equiv \det \gamma_{\alpha\beta} = t^{6\gamma_0} \det a(1 + bt^{2-\gamma_0\alpha_1} + ct^{2-2\gamma_0}$$
$$+ dt^{4-2\gamma_0\alpha_1} + (1/2)(b^2 - b_\alpha^\beta b_\beta^\alpha)t^{4-2\gamma_0\alpha_1}), \tag{35}$$

$$\frac{\dot{\gamma}}{\gamma} = 6\gamma_0 t^{-1} + (2 - \gamma_0\alpha_1)bt^{1-\gamma_0\alpha_1} + 2(1 - \gamma_0)ct^{1-2\gamma_0}$$
$$+ 2(2 - \gamma_0\alpha_1)dt^{3-2\gamma_0\alpha_1} - (2 - \gamma_0\alpha_1)b_\alpha^\beta b_\beta^\alpha t^{3-2\gamma_0\alpha_1}. \tag{36}$$

Now, using well-known expressions for the components of the Ricci tensor [17]:

$$R_0^0 = -\frac{1}{2}\frac{\partial \kappa_\alpha^\alpha}{\partial t} - \frac{1}{4}\kappa_\alpha^\beta \kappa_\beta^\alpha, \tag{37}$$

$$R_\alpha^0 = \frac{1}{2}(\kappa_{\alpha;\beta}^\beta - \kappa_{\beta;\alpha}^\beta), \tag{38}$$

$$R_\alpha^\beta = -P_\alpha^\beta - \frac{1}{2}\frac{\partial \kappa_\alpha^\beta}{\partial t} - \frac{\dot{\gamma}}{4\gamma}\kappa_\alpha^\beta \tag{39}$$

where P_α^β is the three-dimensional part of the Ricci tensor, and substituting the expressions (29)- (36) into Eqs. (37)-(39), one immediately obtains

$$R_0^0 = 3\gamma_0(1 - \gamma_0)t^{-2} - \frac{1}{2}(2 - \gamma_0\alpha_1)(1 - \gamma_0\alpha_1 + 2\gamma_0)bt^{-\gamma_0\alpha_1}$$
$$- (1 - \gamma_0)ct^{-2\gamma_0} - (2 - \gamma_0\alpha_1)(3 - 2\gamma_0\alpha_1 + 2\gamma_0)dt^{2-2\gamma_0\alpha_1}$$
$$+ \frac{1}{4}(2 - \gamma_0\alpha_1)(4 - 3\gamma_0\alpha_1 + 4\gamma_0)b_\alpha^\beta b_\beta^\alpha t^{2-2\gamma_0\alpha_1}, \tag{40}$$

$$R_\alpha^0 = \frac{1}{2}(2 - \gamma_0\alpha_1)(b_{\alpha;\beta}^\beta - b_{;\alpha})t^{1-\gamma_0\alpha_1}$$
$$+ (1 - \gamma_0)(c_{\alpha;\beta}^\beta - c_{;\alpha})t^{1-2\gamma_0} + (2 - \gamma_0\alpha_1)(d_{\alpha;\beta}^\beta - d_{;\alpha})t^{3-2\gamma_0\alpha_1}$$
$$+ \frac{1}{2}(2 - \gamma_0\alpha_1)((b_\nu^\mu b_\nu^\mu)_{;\alpha} - (b_\alpha^\mu b_\mu^\beta)_{;\beta})t^{3-2\gamma_0\alpha_1}. \tag{41}$$

To write down an explicit expression for R_α^β, we need an expression for P_α^β taken up to linear terms in $b_{\alpha\beta}$. It reads

$$P_\alpha^\beta = \tilde{P}_\alpha^\beta t^{-2\gamma_0} - b^{\mu\nu}\tilde{P}_{\mu\alpha\nu}^\beta t^{2-\gamma_0\alpha_1-2\gamma_0}$$
$$+ \frac{1}{2}(2b_{;\nu\alpha}^{\nu\beta} - b_{;}{}^{\beta}{}_{\alpha} - b_{;\alpha\nu}^{\nu\beta} - b_{\alpha;\nu}^{\beta\ \nu} + b_{\alpha;}{}^{\nu\ \beta}{}_\nu)t^{2-\gamma_0\alpha_1-2\gamma_0}. \tag{42}$$

Here, $\tilde{P}_{\mu\alpha\nu}^\beta$ denotes the three-dimensional Riemann-Christoffel tensor for the spatial metric $a_{\alpha\beta}$. Covariant derivatives are also taken with respect to this metric. Now

$$R_\alpha^\beta = -\tilde{P}_\alpha^\beta t^{-2\gamma_0} + b^{\mu\nu}\tilde{P}_{\mu\alpha\nu}^\beta t^{2-\gamma_0\alpha_1-2\gamma_0}$$
$$- \frac{1}{2}(2b_{;\nu\alpha}^{\nu\beta} - b_{;}{}^{\beta}{}_{\alpha} - b_{;\alpha\nu}^{\nu\beta} - b_{\alpha;\nu}^{\beta\ \nu} + b_{\alpha;}{}^{\nu\ \beta}{}_\nu)t^{2-\gamma_0\alpha_1-2\gamma_0}$$
$$+ \gamma_0(1 - 3\gamma_0)\delta_\alpha^\beta t^{-2} - \frac{1}{2}(2 - \gamma_0\alpha_1)((1 - \gamma_0\alpha_1 + 3\gamma_0)b_\alpha^\beta + \gamma_0 b\delta_\alpha^\beta)t^{-\gamma_0\alpha_1}$$
$$- (1 - \gamma_0)((1 - \gamma_0)c_\alpha^\beta + \gamma_0 c\delta_\alpha^\beta)t^{-2\gamma_0}$$

$$-(2-\gamma_0\alpha_1)((3-2\gamma_0\alpha_1+3\gamma_0)d_\alpha^\beta+\gamma_0d\delta_\alpha^\beta)t^{2-2\gamma_0\alpha_1}$$
$$+\frac{1}{2}(2-\gamma_0\alpha_1)\left((3-2\gamma_0\alpha_1+3\gamma_0)b_\alpha^\nu b_\nu^\beta+\gamma_0 b_\mu^\nu b_\nu^\mu\delta_\alpha^\beta\right.$$
$$\left.-\frac{1}{2}(2-\gamma_0\alpha_1)bb_\alpha^\beta\right)t^{2-2\gamma_0\alpha_1}. \tag{43}$$

The Einstein equations are, as usually,

$$R_i^j=8\pi G(T_i^j-\frac{1}{2}\delta_i^j T) \tag{44}$$

where the energy-momentum tensor for a two-fluid model has the form

$$T_i^j=(\varepsilon_1+p_1)u_{1i}u^{1j}-p_1\delta_i^j+(\varepsilon_2+p_2)u_{2i}u^{2j}-p_2\delta_i^j. \tag{45}$$

Analyzing the Friedmann equation (4) again, one can see that series expansions for the energy densities (and, hence, for the pressures) have the following form:

$$\sum_{n,m=0}\varepsilon_1^{(n,m)}(x)t^{-\gamma_0\alpha_1+n(2-2\gamma_0)+m(2-\gamma_0\alpha_1)}, \tag{46}$$

$$\sum_{n,m=0}\varepsilon_2^{(n,m)}(x)t^{-2+n(2-2\gamma_0)+m(2-\gamma_0\alpha_1)}. \tag{47}$$

It is easy to notice that the series expansion (47) contains all powers of t which appear in (46). However, the series (47) contains the term proportional to $1/t^2$ which is absent in the series for ε_1. Besides, the series (47) contains the terms $\varepsilon_2^{(n,0)}t^{-2+n(2-2\gamma_0)}$, $n=1,2,\ldots$ which could appear in the series (46) in the case of the degeneracy of the type given by Eq. (19).

Now let us write down components of the Einstein equations (44) in more detail:

$$R_0^0=4\pi G((1+3k_1)\varepsilon_1-2(1+k_1)\varepsilon_1 u_{1\alpha}u_1^\alpha$$
$$+(1+3k_2)\varepsilon_2-2(1+k_2)\varepsilon_2 u_{1\alpha}u_2^\alpha), \tag{48}$$
$$R_\alpha^0=8\pi G((1+k_1)\varepsilon_1 u_{1\alpha}u_1^0+(1+k_2)\varepsilon_2 u_{2\alpha}u_2^0), \tag{49}$$

$$R_\alpha^\beta=4\pi G(2(1+k_1)\varepsilon_1 u_{1\alpha}u_1^\beta-(1-k_1)\varepsilon_1\delta_\alpha^\beta$$
$$+2(1+k_2)\varepsilon_2 u_{2\alpha}u_2^\beta-(1-k_2)\varepsilon_2\delta_\alpha^\beta). \tag{50}$$

Comparing the structure of the left-hand side of Eq. (49) with its right-hand side, one can see that the series structure for spatial components of fluid four-velocities u_α has the following form:

$$u_{2\alpha}=\sum_{n,m=0}u_{2\alpha}^{(n,m)}t^{1+n(2-\gamma_0)+m(2-\gamma_0\alpha_1)} \tag{51}$$

where $u_{2\alpha}^{(0,0)}=0$, and

$$u_{1\alpha}=\sum_{n,m=0}u_{1\alpha}^{(n,m)}t^{1+n(2-\gamma_0)+m(2-\gamma_0\alpha_1)}. \tag{52}$$

Correspondingly,

$$u_2^\alpha=\sum_{n,m=0}u_2^{\alpha(n,m)}t^{1-2\gamma_0+n(2-\gamma_0)+m(2-\gamma_0\alpha_1)} \tag{53}$$

where $u_2^{\alpha(0,0)} = 0$, and

$$u_1^\alpha = \sum_{n,m=0} u_1^{\alpha(n,m)} t^{1-2\gamma_0+n(2-\gamma_0)+m(2-\gamma_0\alpha_1)}. \tag{54}$$

Also, we need the following expressions:

$$u_{20} = u_2^0 = 1 + \sum_{n,m=0} u_{20}^{(n,m)} t^{2-2\gamma_0+n(2-\gamma_0)+m(2-\gamma_0\alpha_1)} \tag{55}$$

where $u_{20}^{(0,0)} = 0$, and

$$u_{10} = u_1^0 = 1 + \sum_{n,m=0} u_{10}^{(n,m)} t^{2-2\gamma_0+n(2-\gamma_0)+m(2-\gamma_0\alpha_1)}. \tag{56}$$

Now, taking the leading term proportional to t^{-2} in the expansion for R_0^0 and comparing it with the corresponding terms in the series for the energy-momentum tensor, one can find from Eq. (48) that the leading term in the series for the density energy (47) is given by the formula

$$\varepsilon_2^{(0,0)} = \frac{1}{6\pi G(1+k_2)^2}. \tag{57}$$

Thus, the firstly dominant (stiffer) component becomes homogeneous at $t \to 0$.

The next step consists in comparison of the terms proportional to $t^{-2\gamma_0\alpha_1}$ in the equations (48) and (50). As a result, two consistency relations for the two unknown variables $\varepsilon_2^{(0,1)}$ and $\varepsilon_1^{(0,0)}$ arise:

$$- (2 - \gamma_0\alpha_1)(1 - \gamma_0\alpha_1 + 2\gamma_0)b$$
$$= 8\pi G((1 + 3k_1)\varepsilon_1^{(0,0)} + (1 + 3k_2)\varepsilon_2^{(0,0)}, \tag{58}$$

$$(2 - \gamma_0\alpha_1)((1 - \gamma_0\alpha_1 + 3\gamma_0)b_\alpha^\beta + \gamma_0 b\delta_\alpha^\beta)$$
$$= 8\pi G((1 - k_1)\varepsilon_1^{(0,0)}\delta_\alpha^\beta + (1 - k_2)\varepsilon_2^{(0,1)}\delta_\alpha^\beta). \tag{59}$$

It follows from Eq. (59) that

$$b_\alpha^\beta = \frac{1}{3}b\delta_\alpha^\beta. \tag{60}$$

This means that we cannot choose the metric $b_{\alpha\beta}$ freely; it should be proportional to the metric $a_{\alpha\beta}$, i.e.,

$$b_{\alpha\beta} = \frac{1}{3}ba_{\alpha\beta}. \tag{61}$$

Thus, only one new arbitrary function of spatial coordinates $b(\mathbf{r})$ appears in the second term of the series in the right-hand side of Eq. (25). We will immediately see that it describes the amplitude of *isocurvature* perturbations. Really, substituting the expression (61) into Eq. (59), one get

$$(2 - \gamma_0\alpha_1)(1 - \gamma_0\alpha_1 + 6\gamma_0)b$$
$$= 24\pi G((1 - k_1)\varepsilon_1^{(0,0)} + (1 - k_2)\varepsilon_2^{(0,1)}) \tag{62}$$

and then, resolving Eqs. (58) and (62) with respect to $\varepsilon_1^{(0,0)}$ and $\varepsilon_2^{(0,1)}$, we obtain

$$\varepsilon_1^{(0,0)} = \frac{(3k_2 - 2k_1 + 1)b}{12\pi G(k_2 + 1)^2}, \tag{63}$$

$$\varepsilon_2^{(0,1)} = -\frac{b}{12\pi G(k_2 + 1)}. \tag{64}$$

Therefore, the firstly subdominant component 1 is generically essentially inhomogeneous near the singularity.

Now, we should compare terms proportional to $t^{-2\gamma_0}$ in the left- and right-hand sides of Eqs. (48) and (50). We consider a generic situation when the degeneracy described above (see Eq. (13)) is absent. Then we have

$$-(1 - \gamma_0)c = 4\pi G(1 + 3k_2)\varepsilon_2^{(1,0)}, \tag{65}$$

$$\tilde{P}_\alpha^\beta + (1 - \gamma_0)((1 + \gamma_0)c_\alpha^\beta + \gamma_0 c\delta_\alpha^\beta)$$
$$= 4\pi G(1 - k_2)\varepsilon_2^{(1,0)}\delta_\alpha^\beta. \tag{66}$$

Resolving Eqs. (65), (66) with respect to variables c_α^β and $\varepsilon_2^{(1,0)}$, one obtains the following expressions which coincide with known expressions for the one-fluid quasi-isotropic solution [1]:

$$c_\alpha^\beta = -\frac{9(k_2 + 1)^2}{(3k_2 + 1)(3k_2 + 5)}\left(\tilde{P}_\alpha^\beta + \frac{\tilde{P}(3k_2^2 - 6k_2 - 5)\delta_\alpha^\beta}{4(9k_2 + 5)}\right), \tag{67}$$

$$\varepsilon_2^{(1,0)} = \frac{3\tilde{P}(k_2 + 1)}{16\pi G(9k_2 + 5)}. \tag{68}$$

The next step consists in the consideration of terms proportional to $t^{2-2\gamma_0\alpha_1}$. Using Eq. (60), one can obtain from Eqs. (48) and (50) that

$$-(2 - \gamma_0\alpha_1)(3 - 2\gamma_0\alpha_1 + 2\gamma_0)d + \frac{(2 - \gamma_0\alpha_1)(4 - 3\gamma_0\alpha_1 + 4\gamma_0)b^2}{12}$$
$$= 4\pi G(\varepsilon_2^{(0,2)}(1 + 3k_2) + \varepsilon_1^{(0,1)}(1 + 3k_1)), \tag{69}$$

$$(2 - \gamma_0\alpha_1)((3 - 2\gamma_0\alpha_1 + 3\gamma_0)d_\alpha^\beta + \gamma_0 d\delta_\alpha^\beta)$$
$$+ \frac{(2 - \gamma_0\alpha_1)(12 - 7\gamma_0\alpha_1 + 12\gamma_0)b^2\delta_\alpha^\beta}{36}$$
$$= 4\pi G\delta_\alpha^\beta(\varepsilon_1^{(0,1)}(1 - k_1) + \varepsilon_2^{(0,2)}(1 - k_2)). \tag{70}$$

It is easy to see from Eq. (70) that

$$d_\alpha^\beta = \frac{1}{3}d\delta_\alpha^\beta. \tag{71}$$

Substituting Eq. (71) into Eq. (70), one get

$$\frac{(2 - \gamma_0\alpha_1)(3 - 2\gamma_0\alpha_1 + 6\gamma_0)d\delta_\alpha^\beta}{3}$$
$$+ \frac{(2 - \gamma_0\alpha_1)(12 - 7\gamma_0\alpha_1 + 12\gamma_0)b^2\delta_\alpha^\beta}{36}$$
$$= 4\pi G\delta_\alpha^\beta(\varepsilon_1^{(0,1)}(1 - k_1) + \varepsilon_2^{(0,2)}(1 - k_2)). \tag{72}$$

Now, using the definitions of the coefficients α_1, α_2 and γ_0, it is convenient to rewrite Eqs. (69) and (72) in the following form:

$$\varepsilon_2^{(0,2)}(1 + 3k_2) + \varepsilon_1^{(0,1)}(1 + 3k_1)$$
$$= -\frac{(k_2 - k_1)(9k_2 - 12k_1 + 1)d}{6\pi G(k_2 + 1)^2} + \frac{(k_2 - k_1)(6k_2 - 9k_1 + 1)}{36\pi G(k_2 + 1)^2}, \tag{73}$$

$$\varepsilon_2^{(0,2)}(1 - k_2) + \varepsilon_1^{(0,1)}(1 - k_1)$$
$$= \frac{(k_2 - k_1)(3k_2 - 4k_1 + 3)d}{6\pi G(k_2 + 1)^2} + \frac{(k_2 - k_1)(k_1 - 3)}{36\pi G(k_2 + 1)^2}. \tag{74}$$

The system of equations (73) and (74) contains three variables $d, \varepsilon_2^{(0,2)}$ and $\varepsilon_1^{(0,1)}$. To solve it we use the energy conservation law

$$T_{0\ ;i}^{i} = 0. \tag{75}$$

In the case of non-interacting perfect fluids, Eq. (75) gives two separate equations in our approximation:

$$\dot{\varepsilon}_2 + \frac{1}{2}\frac{\dot{\gamma}}{\gamma}(\varepsilon_2 + p_2) = 0; \tag{76}$$

$$\dot{\varepsilon}_1 + \frac{1}{2}\frac{\dot{\gamma}}{\gamma}(\varepsilon_1 + p_1) = 0. \tag{77}$$

Using the series (46), (47) and (36), we get the following formulae connecting $\varepsilon_2^{(0,2)}, \varepsilon_1^{(0,1)}, d$ and b:

$$\varepsilon_2^{(0,2)} = -\frac{d}{12\pi G(1 + k_2)} + \frac{(3k_2 + 5)b^2}{144\pi G(1 + k_2)}, \tag{78}$$

$$\varepsilon_1^{(0,1)} = -\frac{(k_1 + 1)(3k_2 - 2k_1 + 1)b^2}{24\pi G(1 + k_2)^2}. \tag{79}$$

Substituting expressions for the energy densities from Eqs. (78) and (79) into Eq. (73), one get the following expression for d:

$$d = \frac{(-3k_2^2 + 6k_1^2 - 6k_1k_2 - 2k_2 - 2k_1 - 1)b^2}{12(5k_2 - 4k_1 + 1)}. \tag{80}$$

It is easy to check that this expression satisfies Eq. (74), too. Substituting (80) into Eq. (78), we have

$$\varepsilon_2^{(0,2)} = \frac{(3k_2^2 - k_1^2 - k_1k_2 + 5k_2 - 3k_1 + 1)b^2}{24\pi G(1 + k_2)(5k_2 - 4k_1 + 1)}. \tag{81}$$

Now, we would like to write down explicit formulae for first coefficients of the velocity series (51), (52). Using these series and the series for the energy densities (46), (47), one can get the following relation for terms proportional to $t^{1-\gamma_0\alpha_1}$ in the Einstein equation (49):

$$\frac{1}{2}(2 - \gamma_0\alpha_1)(b_{\alpha\ ;\beta}^{\beta} - b_{;\alpha}) = 8\pi G((1 + k_2)\varepsilon_2^{(0,0)}u_{2\alpha}^{(0,1)} + (1 + k_1)\varepsilon_{1\alpha}^{(0,0)}u_{1\alpha}^{(0,0)}). \tag{82}$$

Eq. (82) contains two unknown quantities $u_{2\alpha}^{(0,0)}$ and $u_{1\alpha}^{(0,0)}$. Thus, we need additional relations to resolve it. We will use remaining components of the energy-momentum conservation law:

$$T_{\alpha\ ;i}^i = 0. \tag{83}$$

In the synchronous system of reference, this equation can be rewritten as

$$T_{\alpha\ ,0}^0 + T_{\alpha\ ;\beta}^\beta + \frac{1}{2}k_\beta^\beta T_\alpha^0 = 0. \tag{84}$$

Using the definition of energy-momentum tensor components for perfect fluids and the explicit expression for the trace of the extrinsic curvature (31), we have the following couple of relations in the needed order of perturbation theory:

$$k_2\varepsilon_{2\ ;\alpha}^{(0,1)} = (1+k_2)\frac{\partial}{\partial t}(\varepsilon_2^{(0,0)}u_{2\alpha}^{(0,1)}) + \frac{3\gamma_0}{t}(1+k_2)\varepsilon_2^{(0,0)}u_{2\alpha}^{(0,1)}, \tag{85}$$

$$k_1\varepsilon_{1\ ;\alpha}^{(0,0)} = (1+k_1)\frac{\partial}{\partial t}(\varepsilon_1^{(0,0)}u_{1\alpha}^{(0,0)}) + \frac{3\gamma_0}{t}(1+k_2)\varepsilon_1^{(0,0)}u_{1\alpha}^{(0,0)}. \tag{86}$$

Substituting the expressions (57), (64) and (63) for the coefficients $\varepsilon_2^{(0,0)}, \varepsilon_2^{(0,1)}$ and $\varepsilon_1^{(0,0)}$ into Eqs. (85) and (86), one finds

$$u_{2\alpha}^{(0,1)} = -\frac{k_2(k_2+1)b_{;\alpha}}{2(k_2-2k_1+1)}, \tag{87}$$

$$u_{1\alpha}^{(0,0)} = -\frac{k_1(k_2+1)b_{;\alpha}}{(k_1+1)(k_2-2k_1+1)b}. \tag{88}$$

The direct check shows that the velocity coefficients (87) and (88) satisfy Eq. (82).

The next order term of the quasi-isotropic series solution (proportional to $t^{1-2\gamma_0}$ in the Einstein equation (49)) immediately gives

$$(1-\gamma_0)(c_{\alpha\ ;\beta}^\beta - c_{;\alpha}) = 8\pi G(1+k_2)\varepsilon_2^{(0,0)}u_{2\alpha}^{(1,0)}. \tag{89}$$

Notice that in the absence of degeneracy between exponents of different terms in the perturbation series, this equation contains the contribution from the firstly dominant fluid 2 only. Thus, using the explicit expression for the coefficients c_α^β (see Eq. (67) and the Bianchi identity for the curvature tensor $\tilde{P}_\alpha^\beta$

$$\tilde{P}_{\alpha\ ;\beta}^\beta = \frac{1}{2}\tilde{P}_{;\alpha}\ , \tag{90}$$

one has

$$u_{2\alpha}^{(0,0)} = -\frac{27k_2(k_2+1)^3\tilde{P}_{;\alpha}}{8(3k_2+5)(9k_2+5)}. \tag{91}$$

To find next terms of the quasi-isotropic solution for the velocities, i.e., corrections proportional to $t^{(3-\gamma_0\alpha_1)}$, we shall use the identity (84) again. This leads to the following relations :

$$k_2\varepsilon_{2\ ;\alpha}^{(0,2)} = (3-2\gamma_0\alpha_1)(\varepsilon_2^{(0,0)}u_{2\alpha}^{(0,2)} + \varepsilon_2^{(0,1)}u_{2\alpha}^{(0,1)})$$
$$+ \frac{1}{2}6\gamma_0(\varepsilon_2^{(0,0)}u_{2\alpha}^{(0,2)} + \varepsilon_2^{(0,1)}u_{2\alpha}^{(0,1)})$$
$$+ \frac{1}{2}(2-\gamma_0\alpha_1)b\varepsilon_2^{(0,0)}u_{2\alpha}^{(0,1)}, \tag{92}$$

$$k_1 \varepsilon_1^{(0,1)}{}_{;\alpha} = (3 - 2\gamma_0\alpha_1)(\varepsilon_1^{(0,0)} u_{1\alpha}^{(0,1)} + \varepsilon_1^{(0,1)} u_{1\alpha}^{(0,0)})$$

$$+ \frac{1}{2} 6\gamma_0(\varepsilon_1^{(0,0)} u_{1\alpha}^{(0,1)} + \varepsilon_1^{(0,1)} u_{1\alpha}^{(0,0)})$$

$$+ \frac{1}{2}(2 - \gamma_0\alpha_1) b\varepsilon_1^{(0,0)} u_{1\alpha}^{(0,0)}. \tag{93}$$

From Eqs. (92), (93) one can find the formulae for unknown quantities $u_{2\alpha}^{(0,2)}$ and $u_{1\alpha}^{(0,1)}$ which read

$$u_{2\alpha}^{(0,2)} = \frac{1}{(3k_2 - 4k_1 + 1)\varepsilon_2^{(0,0)}} \times (k_2(k_2 + 1)\varepsilon_2^{(0,2)}{}_{;\alpha}$$

$$- (3k_2 - 4k_1 + 1)\varepsilon_2^{(0,1)} u_{2\alpha}^{(0,1)} - (k_2 - k_1) b\varepsilon_2^{(0,0)} u_{2\alpha}^{(0,1)}), \tag{94}$$

$$u_{1\alpha}^{(0,1)} = \frac{1}{(3k_2 - 4k_1 + 1)\varepsilon_1^{(0,0)}} \times (k_1(k_2 + 1)\varepsilon_1^{(0,1)}{}_{;\alpha}$$

$$- (3k_2 - 4k_1 + 1)\varepsilon_1^{(0,1)} u_{1\alpha}^{(0,0)} - (k_2 - k_1) b\varepsilon_1^{(0,0)} u_{1\alpha}^{(0,0)}). \tag{95}$$

Substituting the explicit expressions for $\varepsilon_2^{(0,0)}, \varepsilon_2^{(0,1)}, \varepsilon_2^{(0,2)}, \varepsilon_1^{(0,0)}, \varepsilon_1^{(0,1)}, u_{2\alpha}^{(0,1)}, u_{1\alpha}^{(0,0)}$ from Eqs. (57), (64), (81), (63), (79), (87), (88) into Eqs. (94) and (95), we get

$$u_{2\alpha}^{(0,2)} = \frac{k_2(k_2 + 1)bb_{;\alpha}}{4(k_2 - 2k_1 + 1)(5k_2 - 4k_1 + 1)(3k_2 - 4k_1 + 1)}$$

$$\times (-9k_2^3 + 18k_1 k_2^2 - 14k_1^2 k_2 + 4k_1^3 + 3k_2^2$$

$$- 6k - 1k_2 + 2k - 1^2 + 5k_2 - 4k - 1 + 1), \tag{96}$$

$$u_{1\alpha}^{(0,1)} = \frac{k_1^2(k_2 + 1)(1 - k_1)(k_2 - 2k_1 - 1)b_{;\alpha}}{2(k_1 + 1)(3k_2 - 4k_1 + 1)(k_2 - 2k_1 + 1)}. \tag{97}$$

In conclusion, let us write down the expressions for the first terms of the quasi-isotropic series solution for the metric, energy densities of two fluids and their 3-velocities calculated above:

$$\gamma_{\alpha\beta} = a_{\alpha\beta} t^{\frac{4}{3(k_2+1)}} + ba_{\alpha\beta} t^{\frac{2(3k_2 - 3k_1 + 2)}{3(k_2+1)}}$$

$$- \frac{9(k_2 + 1)^2}{(3k_2 + 1)(3k_2 + 5)} \left(\tilde{P}_{\alpha\beta} + \frac{(3k_2^2 - 6k_2 - 5)\tilde{P} a_{\alpha\beta}}{4(9k_2 + 5)} \right) t^2$$

$$- \frac{(3k_2^2 - 6k_1^2 + 6k_1 k_2 + 2k_2 + 2k_1 + 1)b^2 a_{\alpha\beta}}{12(5k_2 - 4k_1 + 1)} t^{\frac{2(6k_2 - 6k_1 + 2)}{3(k_2+1)}} + \cdots, \tag{98}$$

$$\varepsilon_2 = \frac{1}{6\pi G(k_2 + 1)^2 t^2} - \frac{b}{12\pi G(k_2 + 1)} t^{-\frac{2(k_1+1)}{k_2+1}}$$

$$+ \frac{3(k_2 + 1)\tilde{P}}{16\pi G(9k_2 + 5)} t^{-\frac{4}{3(k_2+1)}}$$

$$+ \frac{(3k_2^2 - k_1^2 - k_1 k_2 + 5k_2 - 3k_1 + 1)b^2}{24\pi G(k_2 + 1)(5k_2 - 4k_1 + 1)} t^{\frac{2(k_2 - 2k_1 - 1)}{k_2+1}} + \cdots, \tag{99}$$

$$\varepsilon_1 = \frac{(3k_2 - 2k_1 + 1)b}{12\pi G(k_2 + 1)^2} t^{-\frac{2(k_1+1)}{k_2+1}}$$

$$- \frac{(k_1 + 1)(3k_2 - 2k_1 + 1)b^2}{24\pi G(k_2 + 1)^2} t^{\frac{2(k_2 - 2k_1 - 1)}{k_2+1}} + \cdots, \tag{100}$$

$$u_{2\alpha} = -\frac{k_2(k_2+1)b_{;\alpha}}{2(k_2-2k_1+1)}t^{\frac{3k_2-2k_1+1}{k_2+1}}$$

$$-\frac{27k_2(k_2+1)^3\tilde{P}_{;\alpha}}{8(3k_2+5)(9k_2+5)}t^{\frac{9k_2+5}{3(k_2+1)}}$$

$$+\frac{k_2(k_2+1)bb_{;\alpha}}{4(k_2-2k_1+1)(5k_2-4k_1+1)(3k_2-4k_1+1)}$$

$$\times(-9k_2^3+18k_1k_2^2-14k_1^2k_2+4k_1^3+3k_2^2$$

$$-6k_1k_2+2k_1^2+5k_2-4k_1+1)t^{\frac{5k_2-4k_1+1}{k_2+1}}+\cdots, \tag{101}$$

$$u_{1\alpha} = \frac{k_1(k_2+1)b_{;\alpha}}{(k_1+1)(k_2-2k_1+1)b}t$$

$$+\frac{k_1^2(k_2+1)(1-k_1)(k_2-2k_1-1)b_{;\alpha}}{2(k_1+1)(3k_2-4k_1+1)(k_2-2k_1+1)}t^{\frac{3k_2-2k_1+1}{k_2+1}}+\cdots. \tag{102}$$

Note that the velocity flow is potential. Actually, it can be shown that this important property remains in all higher orders of the quasi-isotropic solution. In the degenerate case (see Eq. (13)) when the relation between coefficients k_1 and k_2 is

$$k_1=\frac{3k_2-1}{6}, \tag{103}$$

the terms with superscripts $(1,0)$ and $(0,2)$ in the expressions (98), (99) and (101), i.e., the terms proportional to $\tilde{P}$ and b^2 respectively belong to the same order in the quasi-isotropic series expansion. Then the coefficients at the corresponding powers of time t should be simply added due to linearity of all iterative equations used above.

3. Conclusions and discussion

The equations (98)-(102) represent the main result of the present paper – the first terms in the generalized quasi-isotropic series solution for the case of two barotropic fluids. How many arbitrary physical (i.e., gauge invariant) functions of spatial coordinates does it contain? The answer is four: 3 functions are contained in $a_{\alpha\beta}$ (no conditions are imposed on this symmetric tensor, and the remaining freedom of coordinate transformations at $t=0$ consists of 3 spatial rotations) and the fourth function is b. The former 3 physical functions describe the growing mode of scalar adiabatic perturbations and the non-decreasing mode of gravitational waves (with 2 polarization states) in the regime when metric perturbations are not small, while the latter function describes the growing mode of scalar isocurvature fluctuations.

It is seen that the first term of the generalized quasi-isotropic solution remains factorized with respect to t and spatial coordinates. This represents the generalization of the known result, that small relative metric perturbations in the synchronous system of reference corresponding to the growing adiabatic mode remain constant in the long-wave limit even in a multi-fluid case, to the case when metric perturbations are not small. For the other, isocurvature mode, metric perturbations are zero at $t=0$, but grows with time. However, their back reaction appears in higher order terms of the

generalized quasi-isotropic solution and does not affect the leading term of the adiabatic mode (as far as the series expansion has sense at all).

However, as compared to the one-fluid case, the following new feature arises. In that case, the quasi-isotropic series solution loses sense in the course of future expansion when spatial gradients of all quantities become comparable to temporal ones. The series for the generalized quasi-isotropic solution may diverge for another reason even if spatial derivatives are still small, namely, when higher-order terms containing b become of the order of the leading term. This means that the firstly subdominant fluid becomes dominant and vice versa. Then the law of expansion should change, too (as occurs, e.g., in the standard CDM+radiation FRW model after the moment of matter–radiation equality). This shows that all terms in the generalized quasi-isotropic series containing the function $b(\mathbf{r})$ but not its derivatives can be summed. We leave this question for future work.

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